LIMITES DE FONCTIONS TRIGONOMETRIQUES

EXERCICES 7A.1

Etudier les limites suivantes :

1. $\lim\limits_{x\to 0} \dfrac{\sin\left(x^2\right)}{x}$
2. $\lim\limits_{x\to 0} \dfrac{\cos(3x)-1}{5x}$
3. $\lim\limits_{x\to 0} \dfrac{1-\cos(x)}{2\sin^2(x)}$
4. $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{\sin(5x)}$
5. $\lim\limits_{x\to 4} \dfrac{\sin(2x-8)}{x-4}$
6. $\lim\limits_{x\to 0} \dfrac{3x}{\sin(7x)}$
7. $\lim\limits_{x\to 0} \dfrac{\sin 3x+\sin x}{\sin 5x-\sin x}$
8. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$
9. $\lim\limits_{x\to 0} \dfrac{\cos 2x-\cos 4x}{x^2}$
10. $\lim\limits_{x\to 1} \dfrac{\sin(\pi x)}{x-1}$
11. $\lim\limits_{x\to +\infty} x\sin\left(\dfrac{1}{x}\right)$
12. $\lim\limits_{x\to -\infty} \dfrac{\sqrt{5-\cos x}-2}{x^2}$
13. $\lim\limits_{x\to 0} \dfrac{\cos x-\sqrt{1+\sin^2 x}}{x^2}$

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EXERCICES 7A.1

Etudier les limites suivantes :

1. $\lim\limits_{x\to0}\dfrac{\sin\left(x^2\right)}{x}$ →on pose $X=x^2 \Leftrightarrow x=\sqrt{X}$. Ainsi :

$$\lim_{x\to0}\frac{\sin\left(x^2\right)}{x}=\lim_{X\to0}\frac{\sin(X)}{\sqrt{X}}=\lim_{X\to0}\frac{\sin(X)}{\sqrt{X}}\times\frac{\sqrt{X}}{\sqrt{X}}=\lim_{X\to0}\sqrt{X}\times\frac{\sin(X)}{X}$$

Or $\lim\limits_{X\to0}\sqrt{X}=0$ et $\lim\limits_{X\to0}\dfrac{\sin(X)}{X}=1$, donc par produit des limites : $\lim\limits_{x\to0}\dfrac{\sin\left(x^2\right)}{x}=0$

2. $\lim\limits_{x\to0}\dfrac{\cos(3x)-1}{5x}$ →on pose $X=3x \Leftrightarrow x=\dfrac{1}{3}X$. Ainsi :

$$\lim_{x\to0}\frac{\cos(3x)-1}{5x}=\lim_{X\to0}\frac{\cos(X)-1}{5\times\frac{1}{3}X}=\lim_{X\to0}\frac{3}{5}\frac{\cos(X)-1}{X}$$

Or $\lim\limits_{X\to0}\dfrac{\cos(X)-1}{X}=0$ donc par produit des limites : $\lim\limits_{x\to0}\dfrac{\cos(3x)-1}{5x}=0$

3. $\lim\limits_{x\to0}\dfrac{1-\cos(x)}{2\sin^2(x)}=\lim\limits_{x\to0}\dfrac{1-\cos(x)}{2\left[1-\cos^2(x)\right]}=\lim\limits_{x\to0}\dfrac{1-\cos(x)}{2(1+\cos(x))(1-\cos(x))}=\lim\limits_{x\to0}\dfrac{1}{2(1+\cos(x))}=\dfrac{1}{4}$

4. $\lim\limits_{x\to0}\dfrac{\sin(3x)}{\sin(5x)}$

En posant $X=3x$, on obtient : $\lim\limits_{x\to0}\dfrac{\sin(3x)}{3x}=\lim\limits_{X\to0}\dfrac{\sin(X)}{X}=1$

En posant $X=5x$, on obtient : $\lim\limits_{x\to0}\dfrac{\sin(5x)}{5x}=\lim\limits_{X\to0}\dfrac{\sin(X)}{X}=1$

Ainsi : $\lim\limits_{x\to0}\dfrac{\sin(3x)}{3x}\times\dfrac{5x}{\sin(5x)}\times\dfrac{3}{5}=0,6$, soit : $\lim\limits_{x\to0}\dfrac{\sin(3x)}{\sin(5x)}=0,6$

5. $\lim\limits_{x\to4}\dfrac{\sin(2x-8)}{x-4}$ →on pose $X=2x-8 \Leftrightarrow x-4=\dfrac{1}{2}X$. Ainsi :

$$\lim_{x\to4}\frac{\sin(2x-8)}{x-4}=\lim_{X\to0}\frac{\sin(X)}{\frac{1}{2}X}=2\times\lim_{X\to0}\frac{\sin(X)}{X}=2$$

6. $\lim\limits_{x\to0}\dfrac{3x}{\sin(7x)}$ →on pose $X=7x \Leftrightarrow x=\dfrac{1}{7}X$. Ainsi :

$$\lim_{x\to0}\frac{3x}{\sin(7x)}=\lim_{X\to0}\frac{3\times\frac{1}{7}X}{\sin(X)}=\frac{3}{7}\times\lim_{X\to0}\frac{X}{\sin(X)}=\frac{3}{7}$$

7. $\lim\limits_{x\to0}\dfrac{\sin3x+\sin x}{\sin5x-\sin x}=\lim\limits_{x\to0}\dfrac{\frac{\sin3x+\sin x}{x}}{\frac{\sin5x-\sin x}{x}}=\lim\limits_{x\to0}\dfrac{3\times\frac{\sin3x}{3x}+\frac{\sin x}{x}}{5\times\frac{\sin5x}{5x}-\frac{\sin x}{x}}=...=\dfrac{4}{4}=1$

8. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$ → on part de : $\cos^2 x+\sin^2 x=1$

$$\Leftrightarrow\ \sin^2 x=1^2-\cos^2 x=(1+\cos x)(1-\cos x)$$

$$\Leftrightarrow\ \frac{\sin^2 x}{1+\cos x}=1-\cos x$$

Ainsi : $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\dfrac{\frac{\sin^2 x}{1+\cos x}}{x^2}=\lim\limits_{x\to 0}\dfrac{\sin^2 x}{x^2(1+\cos x)}=\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^2\times\dfrac{1}{1+\cos x}$

Or : $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$, donc :

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{1}{1+\cos x}=\frac{1}{2}$$

9. $\lim\limits_{x\to 0}\dfrac{\cos 2x-\cos 4x}{x^2}=\lim\limits_{x\to 0}\dfrac{\cos 2x-1-\cos 4x+1}{x^2}=\lim\limits_{x\to 0}\dfrac{\cos 2x-1}{x^2}-\dfrac{\cos 4x-1}{x^2}$

Or : $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}=\dfrac{1}{2}$ donc :

$$\lim_{x\to 0}\frac{\cos 2x-1}{x^2}-\frac{\cos 4x-1}{x^2}=\lim_{x\to 0}4\times\frac{\cos 2x-1}{(2x)^2}-16\times\frac{\cos 4x-1}{(4x)^2}$$

On pose $X=2x$ et $Y=4x$

$$\lim_{x\to 0}4\times\frac{\cos 2x-1}{(2x)^2}-16\times\frac{\cos 4x-1}{(4x)^2}=4\times\lim_{X\to 0}\frac{\cos X-1}{X^2}-16\times\lim_{Y\to 0}\frac{\cos Y-1}{Y^2}$$

$$=4\times\left(-\frac{1}{2}\right)-16\times\left(-\frac{1}{2}\right)=-2+8=6$$

10. $\lim\limits_{x\to 1}\dfrac{\sin(\pi x)}{x-1}$ → on pose $X=x-1 \Leftrightarrow x=X+1$. Ainsi :

$$\lim_{x\to 1}\frac{\sin(\pi x)}{x-1}=\lim_{X\to 0}\frac{\sin(\pi(X+1))}{X}=\lim_{X\to 0}\frac{\sin(\pi X+\pi)}{X}=\lim_{X\to 0}\frac{-\sin(\pi X)}{X}$$

$$=\lim_{X\to 0}\frac{-\sin(\pi X)}{\pi X}\times\pi=-1\times\pi=-\pi$$

11. $\lim\limits_{x\to+\infty}x\sin\left(\dfrac{1}{x}\right)$ → on pose $X=\dfrac{1}{x} \Leftrightarrow x=\dfrac{1}{X}$. Ainsi :

$$\lim_{x\to+\infty}x\sin\left(\frac{1}{x}\right)=\lim_{X\to 0}\frac{1}{X}\sin(X)=1$$

12. $\lim\limits_{x\to-\infty}\dfrac{\sqrt{5-\cos x}-2}{x^2}$ → pour tout réel x :

$$-1\le\cos x\le 1$$

$$\Leftrightarrow\ 1\ge-\cos x\ge-1$$

$$\Leftrightarrow\ 6\ge 5-\cos x\ge 4$$

$$\Leftrightarrow\ \sqrt{6}\ge\sqrt{5-\cos x}\ge\sqrt{4}$$

$$\Leftrightarrow\ \sqrt{6}-2\ge\sqrt{5-\cos x}-2\ge 0$$

$$\Leftrightarrow\ \frac{\sqrt{6}-2}{x^2}\ge\frac{\sqrt{5-\cos x}-2}{x^2}\ge\frac{0}{x^2}$$

Or : $\lim\limits_{x\to-\infty}\dfrac{0}{x^2}=\lim\limits_{x\to-\infty}\dfrac{\sqrt{6}-2}{x^2}=0$, donc par encadrement : $\lim\limits_{x\to-\infty}\dfrac{\sqrt{5-\cos x}-2}{x^2}=0$

13. $\lim\limits_{x\to 0}\dfrac{\cos x-\sqrt{1+\sin^2 x}}{x^2}=\lim\limits_{x\to 0}\dfrac{\cos x-1+1-\sqrt{1+\sin^2 x}}{x^2}=\lim\limits_{x\to 0}\dfrac{\cos x-1}{x^2}+\dfrac{1-\sqrt{1+\sin^2 x}}{x^2}\times\dfrac{1+\sqrt{1+\sin^2 x}}{1+\sqrt{1+\sin^2 x}}$

$$=\lim_{x\to 0}\frac{\cos x-1}{x^2}+\frac{1-\left(1+\sin^2 x\right)}{x^2\left(1+\sqrt{1+\sin^2 x}\right)}=\lim_{x\to 0}\frac{\cos x-1}{x^2}+\frac{-\sin^2 x}{x^2\left(1+\sqrt{1+\sin^2 x}\right)}$$

$$=\lim_{x\to 0}\frac{\cos x-1}{x^2}-\left(\frac{\sin x}{x}\right)^2\frac{1}{1+\sqrt{1+\sin^2 x}}$$

$$=-\frac{1}{2}-1\times\frac{1}{1+1}=-1$$